

Econ 802

Lecture Notes on Chapter 7

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Here we start The Theory of The Consumer.

As with The firm we do the following:

- ① describe The objective and The constraints
- ② assume maximizing behavior
- ③ ask about general implications of maximization
- ④ get more specific results for special cases.

Preferences

A consumption bundle is a vector $x = (x_1, \dots, x_n)$ where these are quantities of goods $i = 1 \dots n$.

The consumption set is the set of bundles over which preferences are defined. We normally write it as X (capitalized). Often we have $X = \mathbb{R}_+^n$ so the consumption set is just the set of all non-negative bundles $x = (x_1, \dots, x_n) \geq 0$.

However, there could be exceptions.

For example, we might recognize that there is some lower bound on food consumption below which people will starve or an upper bound on leisure time of 24 hours per day.

We normally assume X is closed and convex.

Preferences are defined using the notation

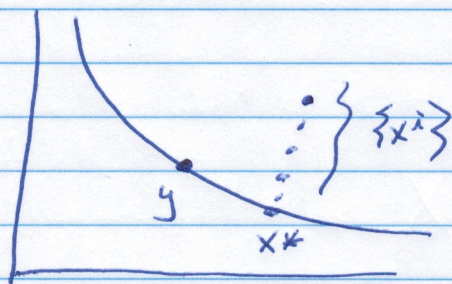
$x \succeq y$ which means "bundle x is at least as good as bundle y ".

Notice That $\succeq$ is not an inequality sign. It is a separate piece of notation referring to preferences.

In this course we make the following standard assumptions about preferences.

- ① Completeness: for any $x, y \in X$ either $x \succeq y$ or $y \succeq x$ or both (any two bundles can be compared)
- ② Reflexivity: for all $x \in X$ we have $x \succeq x$ (any bundle is at least as good as itself)
- ③ Transitivity: for all $x, y, z \in X$ if $x \succeq y$ and $y \succeq z$ then $x \succeq z$ (people are consistent)
- ④ Continuity: for all $y \in X$ the sets $\{x : x \succeq y\}$ and $\{x : x' \preceq y\}$ are closed (each of the sets contains its own boundary)

For example, if we choose some sequence of points $\{x^i\}$ where $x^i \succeq y$ for all i , and $\{x^i\} \rightarrow x^*$ then $x^* \succeq y$.



We haven't defined an indifference curve yet, but intuitively, if there is a point y and an indifference curve going through it, and all $\{x^i\}$ are on or above the indifference

curve, then the limit point x^* must also be on or above the indifference curve (x^* cannot be worse than y)

A little more notation:

Strict Preference: if $x \succeq y$ but $y \succeq x$ does not hold
Then we write $x \succ y$ and say "x is strictly preferred to y"

Indifference: if $x \succeq y$ and $y \succeq x$ both hold,
Then we write $x \sim y$ and say "x and y are indifferent"

The utility function, Consider some function $u(x)$ that assigns real numbers to consumption bundles.

We say $u(x)$ represents the preference ordering $\succeq$ when

$$u(x) \geq u(y) \text{ iff } x \succeq y$$

This means that the number assigned to bundle x is at least as large as the number assigned to y if and only if x is at least as good as y.

It can be shown that such a function exists when the preference ordering $\succeq$ is complete, reflexive, transitive and continuous. Furthermore, the utility function itself will be a continuous function (although it may not be differentiable).

Given one such function $u(x)$, we could equally well use some other function $v(x) = g[u(x)]$ where $g' > 0$ since $v(x)$ will assign the same ordering to the consumption bundles as $u(x)$.

When any increasing monotonic transformation of the utility function is allowed, we call this ordinal utility (only the ordering of the bundles is important - the units in which utility is measured are not).

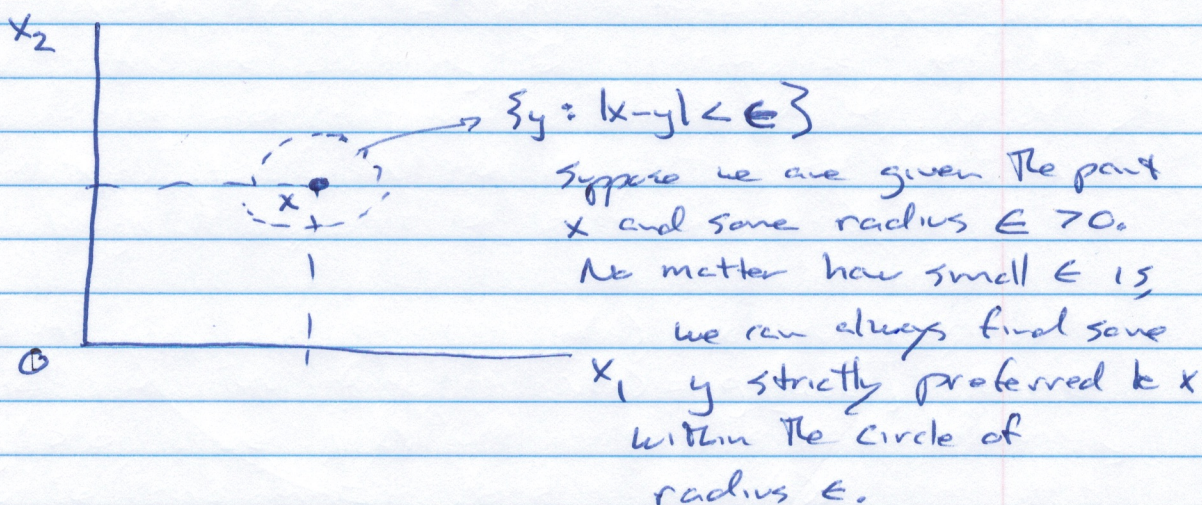
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Other assumptions about preferences are sometimes used:

Weak Monotonicity: if $x \succeq y$ Then $x \succeq y$
(if x has at least as much of all goods as y , Then x is at least as good as y)

Strong Monotonicity: if $x \succeq y$ and $x \neq y$ Then $x \succ y$.
(if x has at least as much of all goods as y , and it has more of at least one good, Then x is strictly preferred to y)

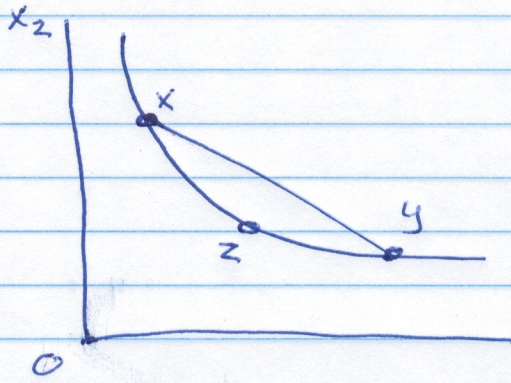
Local Non-Satiation: given any $x \in X$ and any real number $\epsilon > 0$, There is some $y \in X$ with $|x - y| < \epsilon$ such that $y \succ x$.
(for any x , There is some bundle y very close to x such that y is strictly preferred to x).



Convexity: given $x, y, z \in X$ with $x \succeq z$ and $y \succeq z$, any bundle $tx + (1-t)y$ with $0 \leq t \leq 1$ has $tx + (1-t)y \succeq z$.
(if x is in the set of bundles at least as good as z , and y is also in this set, Then so is a convex combination of x and y)

Strict Convexity: given $x, y, z \in X$ with $x \neq y$, if $x \succeq z$ and $y \succeq z$ then $tx + (1-t)y \succ z$ for all $0 < t < 1$.

(If x and y are two distinct points and each is at least as good as z , then any point on the line segment between x and y is strictly better than z)



Again, we haven't defined an indifference curve yet, but here is the general idea of strict convexity. The line segment between x and y lies above the indiff. curve.

Note: strictly convex preferences are equivalent to a strictly quasi-concave utility function. If we have preferences that are convex but not strictly convex, then we have a utility function that is quasi-concave but not strictly quasi-concave.

Indifference Curve (defined by y):

$$\{x \in X : x \succeq y \text{ and } x \preceq y\}$$

So if we have a given bundle y we define the indifference curve through y to be the set of all bundles x that are indifferent to y .

Or, if we have a utility function $u(x)$, we can define the indifference curve as $\{x \in X : u(x) = u(y)\}$

Upper Contour Set (defined by y):

$$\{x \in X : x \succeq y\} \text{ or } \{x \in X : u(x) \geq u(y)\}$$

This is the set of all bundles x that are at least as good as y .

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Note that an indifference curve is similar to an isocost in the theory of the firm, and an upper contour set is similar to an input requirement set.

Consumer Behavior. Assume we have a utility function $u(x)$ representing the consumer's preferences. Our theory says that the consumer solves the following problem:

$$\max_x u(x) \text{ subject to } p \cdot x \leq m \\ x \in X$$

where $p = (p_1, \dots, p_n) > 0$ is the price vector and $m > 0$ is the consumer's income.

Note that the consumer must also choose a bundle in the consumption set X .

An equivalent way of writing this problem is

$$\max_{x \in B} u(x) \quad \text{where } B \equiv \{x \in X : p \cdot x \leq m\}$$

We usually call B the "budget set".

Let's assume $X = \mathbb{R}_+^n$ for simplicity so we only have to think about bundles with $x \geq 0$.

There are several important points about this problem.

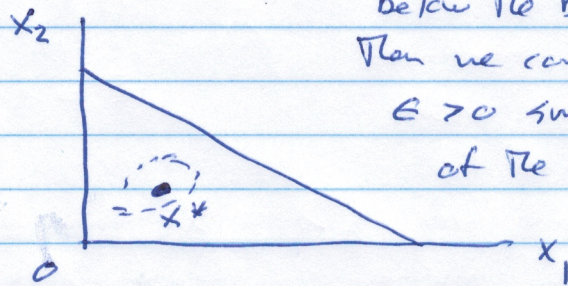
① A solution always exists. This is true because the budget set B is compact (closed and bounded) and the utility function is continuous. So the Weierstrass Theorem says that $u(x)$ must take on a maximum value somewhere in the set B .

② The solution is independent of positive monotonic

transformations of the utility function. As long as the function used also represents the same underlying preferences we get the same most preferred bundle x^* .

- ③ If strict convexity holds, the solution x^* is unique for given p and m .
- ④ The optimal choice is homogeneous of degree zero in (p, m) . Multiplying all of the prices and income by the same $t > 0$ does not change the budget set B so it cannot change the solution x^* .
- ⑤ If local non-satiation holds, then $px^* = m$. (The consumer spends all of their income). This follows because if we have local non-satiation but $px^* < m$, we get a contradiction.

Suppose we have this situation (the solution x^* is below the budget constraint).



Then we can always choose some $\epsilon > 0$ small enough that all of the points y with $|x - y| < \epsilon$ are also below

the budget constraint. Therefore all such points are feasible. But LNS says at least one such y is strictly preferred to x . This contradicts the assumption that x^* is optimal. Therefore, when we have LNS we must also have $px^* = m$ (the solution must be on the budget constraint, not below it).

The indirect utility function

It is often useful to define the function

$$v(p, m) \equiv \max_{x \geq 0} u(x) \text{ subject to } p \cdot x = m$$

Note: I am assuming $X = \mathbb{R}_+^n$ so we only need the non-negativity constraint $x \geq 0$. I am also assuming LNS so we can ignore points with $p \cdot x < m$; these points will never solve the problem.

$v(p, m)$ is called the indirect utility function. It gives the maximum possible level of utility associated with given prices and income (p, m) .

If the utility max problem has a unique solution $x(p, m)$ for each (p, m) , then we can also write

$$v(p, m) = u[x(p, m)]$$

Just substitute the optimal bundle $x(p, m)$ into the direct utility function in order to get the indirect utility function. Later we will call the solution $x(p, m)$ the Marshallian demands.

Now let's think about calculus, FOC, SOC, etc.

The Lagrangean for the utility max problem is

$$L(x, d) = u(x) - d[p \cdot x - m] \quad (\text{note: I am ignoring } x \geq 0 \text{ here})$$

$$\text{The FOC are } \left. \begin{array}{l} \frac{\partial u(x^*)}{\partial x} = dp \\ p x^* = m \end{array} \right\} \text{ where } \frac{\partial u}{\partial x} = \left[\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right] \text{ and } p = (p_1, \dots, p_n)$$

From the FOC we have $\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i$
 and $\frac{\partial u(x^*)}{\partial x_j} = \lambda p_j$ } for all i, j

Divide the first equation by the second to get

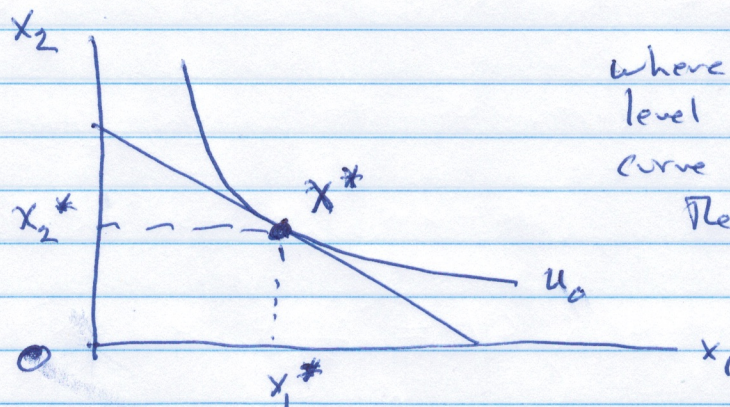
$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} = \frac{p_i}{p_j}$$

It can be shown that the slope of the indifference curve through the point x^* is $-\frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}}$ (the negative of the ratio of the marginal utilities).

The proof is the same as when we showed that the slope of the isoquant through an input vector was equal to the negative of the ratio of the marginal products.

Furthermore, the budget constraint has $p_1 x_1 + p_2 x_2 = m$
 or $x_2 = \frac{m - p_1 x_1}{p_2}$ so the slope of the budget line is $-\frac{p_1}{p_2}$.

Putting this together we must have a tangency point between the indiff. curve and the budget line:



where u_0 is the utility level for the indifference curve passing through the solution x^* .

(10)

On p. 9 I assumed the second order conditions were satisfied and just looked at first order conditions.

Now let's think about the SOC for the Lagrangean $L(x, \lambda) = u(x) - \lambda(p \cdot x - m)$

The necessary SOC says

$$h \frac{\partial^2 L(x^*)}{\partial x^2} h \leq 0 \text{ for all } h \text{ such that } p \cdot h = 0$$

(Think about h as a deviation from x^* ; since $p \cdot x^* = m$, and we need to have $x^* + h$ satisfy the budget constraint we need $p \cdot (x^* + h) = m \Rightarrow p \cdot x^* + p \cdot h = m \Rightarrow p \cdot h = 0$.)

This says that the Hessian of the utility function must be negative semi-definite for all deviations from x^* that satisfy the budget constraint.

The sufficient SOC says

$$h \frac{\partial^2 L(x^*)}{\partial x^2} h < 0 \text{ for all } h \neq 0 \text{ such that } p \cdot h = 0.$$

I would like to dig a little more deeply into the reasons for these conditions. Suppose x^* satisfies the FOC and expand the utility function around x^* . This gives

$$u(x) = u(x^*) + \frac{\partial u(x^*)}{\partial x} (x - x^*) + \frac{1}{2} (x - x^*) \frac{\partial^2 u(x^*)}{\partial x^2} (x - x^*) + \underbrace{\alpha (x - x^*)^3}_{\text{terms of order 3 or higher}}$$

Now write $h = x - x^*$ and consider $p \cdot x = p \cdot x^* = m$. This implies $p \cdot h = 0$ (we are constrained by the budget not to spend more than m , and due to local non-satiation it will never be optimal to spend less).

In order for x^* to be a maximum, we need

$$u(x) - u(x^*) \leq 0 \text{ for all } h \text{ such that } p \cdot h = 0$$

Using the expansion at the bottom of p. 10, this gives

$$\frac{\partial u(x^*)}{\partial x} \cdot h + \frac{1}{2} h' \frac{\partial^2 u(x^*)}{\partial x^2} h + \underbrace{O(h^3)}_{\substack{\text{higher order terms} \\ \text{can be ignored}}} \leq 0$$

From FOC $\frac{\partial u(x^*)}{\partial x} = dp$ when h is small.

and we are assuming $p \cdot h = 0$ anyway so

$\frac{\partial u(x^*)}{\partial x} \cdot h = 0$. So the first order term drops out.

This leaves the necessary SOC

$$h' \frac{\partial^2 u(x^*)}{\partial x^2} h \leq 0 \text{ for all } h \text{ with } p \cdot h = 0$$

(This is necessary because if it did not hold, we could find some deviation h that satisfies the budget constraint and gives $u(x) > u(x^*)$ so x^* is not a solution).

This can be strengthened to local sufficiency if

$$< 0 \text{ holds for all } h \neq 0 \text{ with } p \cdot h = 0$$

(This is sufficient because there is some small neighborhood of x^* where it is impossible to deviate in a way that gives $u(x) > u(x^*)$).

It is important to recognize that the preceding SOC are local. We are evaluating derivatives at a suspected solution x^* and checking whether it is possible to increase utility in a small neighborhood of x^* . But sometimes we have global information.

If the utility function is globally quasi-concave then a point satisfying FOC is a solution (although maybe not unique).

If the utility function is globally strictly quasi-concave then a point satisfying FOC is the unique solution.

In these cases, there is no need to check the SOC in order to see whether x^* is a solution. The FOC will be sufficient for this.

Properties of the Indirect Utility Function

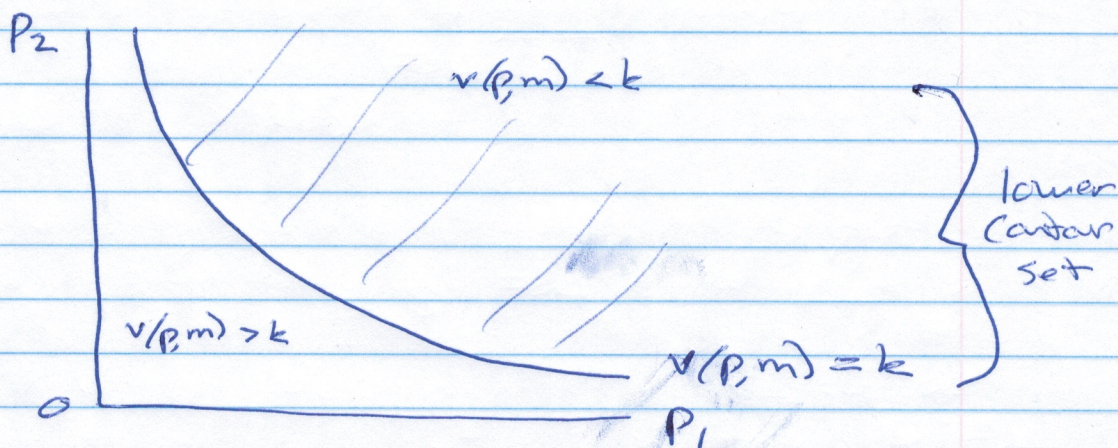
The indirect utility function $v(p, m)$ has four general properties that do not depend on calculus.

- ① Non-increasing in p , non-decreasing in m .
(This follows from the fact that increasing a price never expands the budget set, so utility cannot rise, and increasing income never shrinks the budget set, so utility cannot fall).
- ② Homogeneous of degree zero in (p, m) .
(This follows from the fact that multiplying prices and income by the same $t > 0$ never changes the budget set).

③ Quasi-convex in p .

The indirect utility function is quasi-convex in p if the set $\{p > 0 : v(p, m) \leq k\}$ is a convex set for all real numbers k .

To see this graphically, put the prices on the axes and think about an indifference curve:



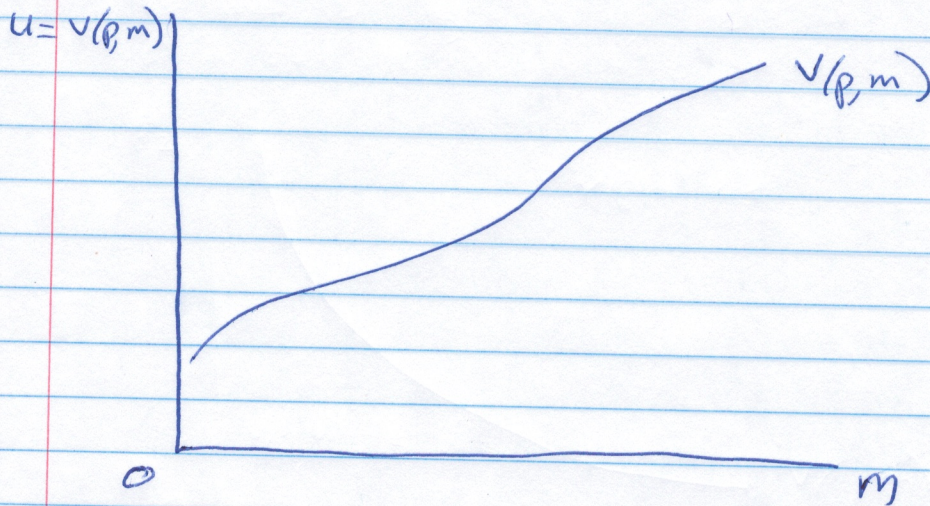
When we are working with the indirect utility function $v(p, m)$ for a fixed income m , it will usually be true that raising a price makes the consumer worse off, so in price space, indifference curves closer to the origin are better. Now define the lower contour set $\{p > 0 : v(p, m) \leq k\}$. It can be shown that this is a convex set, and mathematically this means $v(p, m)$ is quasi-convex as a function of p .

④ Continuous in p and m for $p > 0, m > 0$.

This follows from the Theorem of the Maximum.

Next, fix the prices p and think about variations in income m .

If we have local non-satiation then $v(p, m)$ is strictly increasing in m . So we get



Note: The slope of this curve is not meaningful because we have ordinal utility and we could always take a positive monotonic transformation of the previous direct utility function $u(x)$ which would generally change the slope of $v(p, m)$. For example we could take the log of $u(x)$, the cube of $u(x)$ etc. [Be careful about squaring it though - if the utility numbers can be negative, this will change the ranking of the consumption bundles]

For similar reasons we can always add or subtract a constant to a utility function, multiply it by a positive constant, etc. Thus the intercepts are not meaningful (so $v(p, m)$ doesn't have to go through the origin). All we really know is that $v(p, m)$ increases as m increases.

The expenditure function

Define $e(p, u) \equiv \min_x p \cdot x$ subject to $u(x) \geq u$.

This is often a convenient function in consumer theory; it gives the minimum expenditure required for the consumer to achieve a given utility level u . (we are assuming the prices p are fixed)

This is exactly analogous to the cost function in the theory of the firm: we are just replacing the production function by a utility function.

Therefore, the expenditure function has all the same properties as a cost function:

- ① Non-decreasing in p
- ② Homogeneous of degree 1 in p
- ③ Concave in p
- ④ Continuous in p for $p > 0$

We also get Shephard's Lemma:

$$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u) \quad \text{all } i = 1 \dots n$$

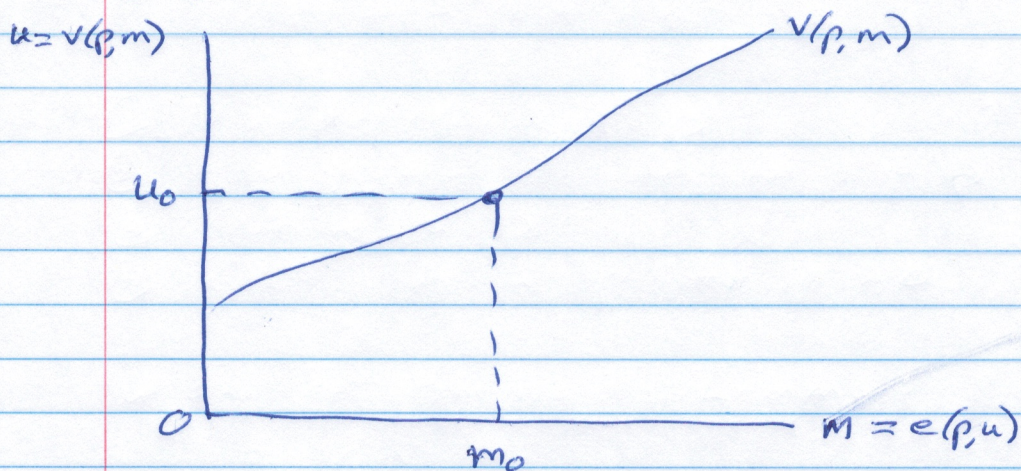
Where $h_i(p, u)$ is the amount of x_i used to minimize the expenditure needed in attaining u . (the solution of the cost-min problem).

For a firm, we called this the conditional demand for x_i . Here we call it the Hicksian or compensated demand function for x_i .

So an important distinction is that when we are maximizing the (direct) utility function subject to a budget constraint, we write the solutions as $x_i(p, m)$, $i = 1 \dots n$, and call them Marshallian demands.

But when we are minimizing expenditure subject to a constraint on (direct) utility, we write the solutions as $h_i(p, u)$, $i = 1 \dots n$, and call them Hicksian demands.

Graphically the expenditure function is the inverse of the indirect utility function (holding prices p fixed). To see this, think about the previous graph:

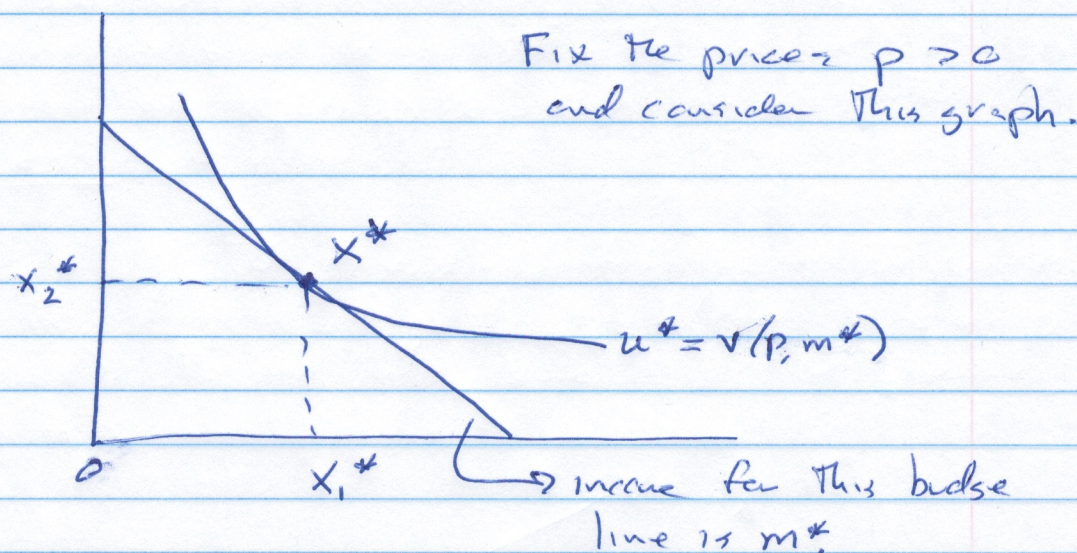


Going from the horizontal axis, if we have income m_0 then the maximum utility we can get is $u_0 = v(p, m_0)$.

But if we start from the vertical axis and require the utility level u_0 , the minimum expenditure needed to achieve u_0 is m_0 , and $e(p, u_0) = m_0$.

We can develop this relationship between $v(p, m)$ and $e(p, u)$ further by using the idea of duality.

Suppose x^* solves a utility max problem. Then it simultaneously solves an expenditure min problem.



Assuming we have local non-satiation, so x^* is on the budget line, then the following identities hold:

$$(A) \quad u^* \equiv v(p, m^*) = \max_x u(x) \text{ subject to } px \leq m^*$$

$$(B) \quad m^* \equiv e(p, u^*) = \min_x p \cdot x \text{ subject to } u(x) \geq u^*.$$

These results imply

$$\left. \begin{array}{l} (1) \quad m^* \equiv e(p, v(p, m^*)) \\ (2) \quad u^* \equiv v(p, e(p, u^*)) \end{array} \right\} \begin{array}{l} \text{just substitute (B)} \\ \text{into (A) or vice versa.} \end{array}$$

Furthermore the fact that x^* solves both problems means that $x_i^* \equiv x_i(p, m^*) \equiv h_i(p, u^*)$. Therefore

$$\left. \begin{array}{l} (3) \quad h_i[p, v(p, m^*)] \equiv x_i(p, m^*) \\ (4) \quad x_i[p, e(p, u^*)] \equiv h_i(p, u^*) \end{array} \right\} \begin{array}{l} \text{again just} \\ \text{substitute from} \\ \text{(A) or (B)} \end{array}$$

Roy's Identity

We can use the preceding duality relationships to compute the Marshallian demand functions using the indirect utility function.

$$\text{The method is} \quad x_i(p, m) = - \frac{\frac{\partial v(p, m)}{\partial p_i}}{\frac{\partial v(p, m)}{\partial m}} \quad \text{for } i=1 \dots n$$

(assuming the derivatives exist)

Proof: Choose arbitrary prices $p^* > 0$ and income $m^* > 0$. Let x^* maximize utility at (p^*, m^*) so $x^* = x(p^*, m^*)$. Write $u^* = u[x(p^*, m^*)] = v(p^*, m^*)$.

We know that $u^* \equiv v[p, e(p, u^*)]$ for any p (including p^*). This follows from identity (4) on p. 17.

We can differentiate this identity with respect to p_i and evaluate all derivatives at (p^*, m^*) :

$$0 = \frac{\partial v(p^*, m^*)}{\partial p_i} + \frac{\partial v(p^*, m^*)}{\partial m} \frac{\partial e(p^*, u^*)}{\partial p_i}$$

Note: derivatives can be evaluated at (p^*, m^*) because $m^* = e(p^*, u^*)$.

Now use Shephard's Lemma to get

$$\frac{\partial e(p^*, u^*)}{\partial p_i} = \underbrace{h_i(p^*, u^*)}_{\text{duality}} = x_i(p^*, m^*)$$

Therefore

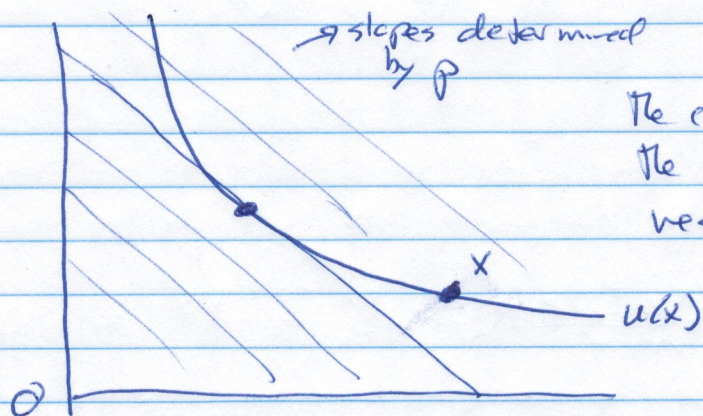
$$x_i(p^*, m^*) = \frac{\partial e(p^*, u^*)}{\partial p_i} = - \frac{\frac{\partial v(p^*, m^*)}{\partial p_i}}{\frac{\partial v(p^*, m^*)}{\partial m}}$$

But p^* and m^* were chosen arbitrarily, so this is true for any (p, m) , and we have

$$x_i(p, m) = - \frac{\frac{\partial v(p, m)}{\partial p_i}}{\frac{\partial v(p, m)}{\partial m}} \quad \text{for all } i=1 \dots n$$

Money Metric Utility Functions

Sometimes in applied work, we have a question like this: given that the consumer has goods x , how much money income would the consumer need at prices p to be as well off as with the consumption bundle x ?



The answer is:

The consumer needs to have the income such that the resulting budget line achieves utility $u(x)$.

But we know that the minimum expenditure necessary to achieve $u(x)$ is

$$m = e(p, u(x))$$

More generally, we can write the "direct money metric" utility function as

$$m(p, x) \equiv e(p, u(x))$$

This gives the level of money income needed to achieve utility $u(x)$ when the prices are p .

For any fixed $p > 0$, this is a utility function defined on x . In fact, it is a positive monotonic transformation of $u(x)$, because $\frac{\partial e}{\partial u} > 0$ (if $u(x)$ assigns a higher utility number to x than y , the same is true for $m(p, x)$).

So there is no reason why we can't use $m(p, x)$ to represent the same preferences as $u(x)$.

But $m(p, x)$ is measured in money units, which is sometimes convenient.

And: $m(p, x)$ behaves like an expenditure function with respect to p .

Another type of question that sometimes comes up is: Suppose the consumer initially faces prices q and has income m . But now the prices change to p . What income does the consumer need now in order to be as well off as before?

The consumer's initial level of utility was $v(q, m)$ from the indirect utility function.

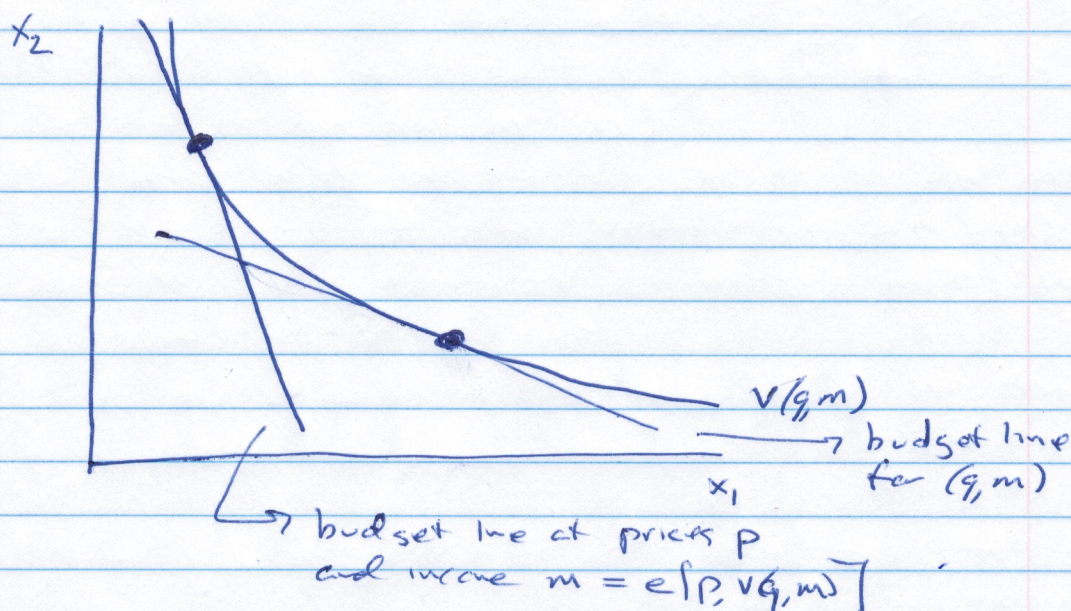
So the question is: given the new prices p , how much income does the consumer need to achieve the utility $v(q, m)$?

Again we use the expenditure function to find the answer: the minimum amount of income required at prices p to achieve utility $v(q, m)$ is

$$m = e(p, v(q, m))$$

Varian writes this as $\mu(p; q, m) \equiv e(p, v(q, m))$

Graphically, here is the idea:



The function $\mu(p; q, m)$ behaves like an expenditure function with respect to p .

It is also an indirect utility function with respect to (q, m) because it is a pos. monotone transformation of the indirect utility function $v(q, m)$.

That's all for Chapter 7!